УДК 510.644
AMS MSC2020: 03C13

In defense of the self-reference quantifier Sx. Approximation by dynamic systems.

Stepanov V. A.

Dorodnitsyn Computing Center of FIC CSC RAS

Abstract. Arguments in defense of introducing the self-referencing quantifier Sx and its approximation on dynamical systems are consistently presented. The case of classical logic is described in detail. Generated 3-valued truth tables that match the corresponding Priest tables [5]. In the process of constructing 4-valued truth tables, two more truth values were revealed that did not coincide with the original ones. Therefore, the closed tables turned out to be 6-valued. De Morgan's law confirmed in 6-valued truth tables.

Keywords: self-reference quantifier Sx, dynamic systems, truth table, Liar, TruthTeller.

Introduction

We are talking about the S icon, which first appeared in the article [4]: $Q =_{df} S_Q P$. According to the meaning, S indicates that the entire expression belongs to self-referencing, and introduces the entire self-referential construction to the rank of WFF. The Liar sentence: $S_Q \sim TQ$.

1. Basic definitions

Self-referential sentences deserve to be marked out in language for their self-referencing. To do this, we fix the self-referencing of the sentence using a special icon — the self-referencing icon Sx, which is placed in front of the predicate $P(x)$, which we call the core of the self-referential sentence. As a result, a self-referential sentence looks like this:

$$\mathrm{S}xP(x). \tag{1}$$

In place of the variable x in $P(x)$ from (1), nothing can be substituted except for this sentence itself. You cannot substitute anything in the newly received sentence in x, except for this sentence itself, etc. Those. a self-referential sentence is outwardly closed, and the expression Sx, according to this criterion, can well be attributed to quantifiers, because it is the presence of Sx that makes expression (1) closed. Expression (1) obeys the axiom of self-reference, which is the essence of the axiom of a fixed point, [3]:

$$\mathrm{S}xP(x) = P(\mathrm{S}xP(x)) \tag{2}$$

Peirce [2] intuitively applied (2) to generate an infinite Liar sentence:

$$\mathrm{S}xP(x) = P(P(P(P(\ldots \mathrm{S}xP(x)\ldots)))) \tag{3}$$

This infinite sentence consists of an infinite number of nested Liar kernels. Let's break it down into iterative steps, discarding the "last" expression $\ldots \mathrm{S}xP(x)\ldots$

$$\mathrm{S}xP(x) \approx \mathsf{S}xP(x) \rightleftharpoons \langle x, P(x), P(P(x)), P(P(P(x))), \ldots\rangle \tag{4}$$

The $\approx$ indicates an approximation. Expression $\mathsf{S}xP(x)$ in (4) on the right will be considered as an approximation of a real self-referential sentence $\mathrm{S}xP(x)$. To denote the result of the approximation, we will choose the sign S to distinguish it from S — a real quantifier of self-reference. The expression $\mathsf{S}x$ will also be called a self-referencing quantifier, if this does not lead to an error.

In front of the sequence of kernels in (4), we insert the variable $x = P^0(x)$ to distinguish one specific branch of the approximation from another. Expression (4) is the definition of the trajectory of a dynamical system of the form $(\{0,1\}, P(x))$ with orbits $\langle P^n(x), n \in Z^+\rangle$, where $P^n(x) = P(P^{n-1}(x))$. This justifies the title of our article. Expression (4) in the theory of dynamical systems [1] is called the trajectory or orbit of the dynamical system. We use the characteristics of such a movement here. Consider the case when the kernels of self-referential sentences $P(x)$ are composed of $Tr(x)$ using propositional connectives $\leftrightarrow$ and $\neg$:

$$P(x) \in \{Tr(x), \neg Tr(x), Tr(x) \leftrightarrow Tr(x), Tr(x) \leftrightarrow \neg Tr(x)\}. \tag{5}$$

The rest of the formulas we are considering are equivalent to these four. The variable$x$ and the predicates $P(x)$ from (5) in our case take values from $\{0,1\}$. It is easy to see that expression (4) is periodic, with a maximum period of 2. This means that the second and third terms of the sequence (4) determine the entire remaining infinite sequence. Therefore, in our case, we rightfully shorten the definition of a self-referencing quantifier as follows:

$$\mathsf{S}xP(x) \rightleftharpoons \langle x, P(x), P(P(x))\rangle. \tag{6}$$

Since there are only two values of x in sequence (6) in our case: $x \in \{0,1\}$, then statement (6) itself splits into two sequences. And since we have no reason to give preference to any one of them, we will combine them as equal rights elements of the set in (7):

$$\mathsf{S}xP(x) = \{\langle 1, P(1), P(P(1))\rangle, \langle 0, P(0), P(P(0)\rangle\}. \tag{7}$$

In the case when the values of x will be more (or less) than two, the number of members of the sets in (6) and (7) should be changed accordingly. This is one of the properties of the definition of the approximation of the self-referencing quantifier $\mathsf{S}x$, which allows it to be used in other logical systems, and not only in classical ones, as in the case under consideration. Now let us define the action of the external negation sign $\neg$. To do this, we will divide our manipulations into several cases. The first of them is when the kernel $P(x)$ of a self-referential sentence is the identically true:
$[P(x) = (Tr(x) \leftrightarrow Tr(x))$, i. e. $P(0) = P(1) = 1]$ or the identically false: $[P(x) = (Tr(x) \leftrightarrow \neg Tr(x))$, i. e. $P(0) = P(1) = 0]$ formula. Then, for example, for $P(x) = 1$ we get

$$\begin{aligned}
\neg\mathsf{S}xP(x) &= \neg\{\langle 1,1,1\rangle, \langle 0,1,1\rangle\} && (= \neg T)\\
&= \{\neg\langle 1,1,1\rangle, \neg\langle 0,1,1\rangle\}\\
&= \{\langle \neg 1,\neg 1,\neg 1\rangle, \langle \neg 0,\neg 1,\neg 1\rangle\}\\
&= \{\langle 0,0,0\rangle, \langle 1,0,0\rangle\} && (= F).
\end{aligned}$$

In the case of nonidentical formulas, $Tr(x)$ (TruthTeller) or $\neg Tr(x)$ (Liar), the estimate of the formula changes along with the estimate for the free variable x:

$$\begin{aligned}
\neg\mathsf{S}xP(x) &= \neg\{\langle 1,0,1\rangle, \langle 0,1,0\rangle\}\\
&= \{\langle \neg 1,\neg 0,\neg 1\rangle, \langle \neg 0,\neg 1,\neg 0\rangle\}\\
&= \{\langle 0,1,0\rangle, \langle 1,0,1\rangle\}\\
&= \{\langle 1,0,1\rangle, \langle 0,1,0\rangle\} && (= \mathsf{S}xP(x))
\end{aligned}$$

This is the table for the negation symbol:

$\mathsf{S}xP(x)$	$\neg\mathsf{S}xP(x)$
$\{\langle 1,1,1\rangle;\langle 0,1,1\rangle\}=T$	$F=\{\langle 1,0,0\rangle;\langle 0,0,0\rangle\}$ (False)
$\{\langle 1,0,1\rangle;\langle 0,1,0\rangle\}=A$	$A=\{\langle 0,1,0\rangle;\langle 1,0,1\rangle\}$ (Antinomy)
$\{\langle 1,1,1\rangle;\langle 0,0,0\rangle\}=V$	$V=\{\langle 0,0,0\rangle;\langle 1,1,1\rangle\}$ (Void)
$\{\langle 1,0,0\rangle;\langle 0,0,0\rangle\}=F$	$T=\{\langle 1,0,0\rangle;\langle 0,0,0\rangle\}$ (True)

We define two-place connectives $\circ\in\{\wedge,\vee,\to,\leftarrow,\leftrightarrow\}$ for two S-formulas $\mathsf{S}xP(x)$ and $\mathsf{S}xQ(x)$. We study such a variant of two-place connectives, when the trajectories of estimates of the formula $\mathsf{S}xP(x)$ of the one branch ($x=1$ or $x=0$) interact with the trajectories of the formula $\mathsf{S}xQ(x)$ of the same branch ($x=1$ or $x=0$):

$$\langle 1,P(1),P(P(1))\rangle\circ\langle 1,Q(1),Q(Q(1))\rangle,$$
$$\langle 0,P(0),P(P(0)\rangle\circ\langle 0,Q(0),Q(Q(0))\rangle:$$

$$\mathsf{S}xP(x)\circ\mathsf{S}xQ(x)=$$
$$\{\langle 1,P(1),P(P(1))\rangle,\langle 0,P(0),P(P(0)\rangle\}\circ$$
$$\{\langle 1,Q(1),Q(Q(1))\rangle,\langle 0,Q(0),Q(Q(0))\rangle\}=$$
$$\{\langle 1,P(1),P(P(1))\rangle\circ\langle 1,Q(1),Q(Q(1))\rangle,$$
$$\langle 0,P(0),P(P(0))\rangle\circ\langle 0,Q(0),Q(Q(0))\rangle\}=$$
$$\{\langle 1\circ 1,P(1)\circ Q(1),P(P(1))\circ Q(Q(1))\rangle,$$
$$\langle 0\circ 0,P(0)\circ Q(0),P(P(0))\circ Q(Q(0))\rangle\}.$$

Here are examples of the interactions between the estimates of *Liar* A (and *TruthTeller* V) with T, F:

$$\mathrm{V}\wedge V=\{\langle 1,1,1\rangle,\langle 0,1,1\rangle\}\wedge\{\langle 1,1,1\rangle,\langle 0,0,0\rangle\}=$$
$$=\{\langle 1,1,1\rangle,\langle 0,0,0\rangle\}=\mathrm{V}$$
$$\mathrm{A}\wedge A=\{\langle 1,1,1\rangle,\langle 0,1,1\rangle\}\wedge\{\langle 1,0,1\rangle,\langle 0,1,0\rangle\}=$$
$$=\{\langle 1,0,1\rangle,\langle 0,1,0\rangle\}=\mathrm{A}$$
$$\mathrm{F}\wedge V=\{\langle 1,0,0\rangle,\langle 0,0,0\rangle\}\wedge\{\langle 1,1,1\rangle,\langle 0,0,0\rangle\}=$$
$$=\{\langle 1,0,0\rangle,\langle 0,0,0\rangle\}=\mathrm{F}$$
$$\mathrm{F}\wedge A=\{\langle 1,0,0\rangle,\langle 0,0,0\rangle\}\wedge\{\langle 1,0,1\rangle,\langle 0,1,0\rangle\}=$$
$$=\{\langle 1,0,0\rangle,\langle 0,0,0\rangle\}=\mathrm{F}$$

2. Main results

Let's reproduce Priest's tables and compare them with ours, built on our rules: V and A

Hypothesis: p = V

$\wedge$	T	V	F
T	T	V	F
V	V	V	F
F	F	F	F

Priest p

$\wedge$	t	p	f
t	t	p	f
p	p	p	f
f	f	f	f

Hypothesis: p = A

$\wedge$	T	A	F
T	T	A	F
A	A	A	F
F	F	F	F

Comparing our table for A (*Liar*) with Priest's table for p (*Liar*) in [5], we notice that they are identical. It should be borne in mind that our tables are built on a completely different principle, different from the principles of Priest's construction. And this inspires a certain optimism, when two completely different principles of construction, so to speak, "external" (priest's) and "internal" (ours), lead to the same result. Comparing our table for V (*TruthTeller*), with Priest's table for p (*Liar*) in [5], we notice that they have the same configuration. But Priest, in his work [5], considers only the *Liar* sentence. Therefore, we will build four-valued tables in which our A and V will be able to interact, with their different truth estimates.

$\wedge$	T	A	V	F
T	T	A	V	F
A	A	A	av	F
V	V	av	V	F
F	F	F	F	F

$\vee$	T	A	V	F
T	T	T	T	T
A	T	A	va	A
V	T	va	V	V
F	T	A	V	F

Here new assessments from interaction appear A and V: va and av.

$$\mathrm{A} \wedge \mathrm{V} = \{\langle 1,0,1\rangle, \langle 0,0,0\rangle\} = \mathrm{av}$$

$$\mathrm{A} \vee \mathrm{V} = \{\langle 1,1,1\rangle, \langle 0,1,0\rangle\} = \mathrm{va}$$

Closed value tables will look like this:

$\wedge$	T	av	A	V	va	F
T	T	av	A	V	va	F
av	av	av	av	av	av	F
A	A	av	A	av	A	F
V	V	av	av	V	V	F
va	va	av	A	V	va	F
F	F	F	F	F	F	F

$\vee$	T	av	A	V	va	F
T	T	T	T	T	T	T
av	T	av	A	V	va	av
A	T	A	A	va	va	A
V	T	V	va	V	va	V
va	T	va	va	va	va	va
F	T	av	A	V	va	F

This allows us to prove the following lemma:

LEMMA 1. 1) *The sentences Liar (A) have the tabular model, coinciding with tabular model Liar (p) of Priest [5] and, accordingly, the same evidential theory.*

2) *The sentences TruthTeller (V) have the same configuration tabular model, coinciding with configuration tabular model Liar (p) of Priest [5].*

3) *When constructing truth tables for the interaction of V and A, new truth values were obtained: V∧A = av and V∨A = va; ¬av=va; ¬va=av.*

In the same way, we will construct tables for disjunction, implication and reverse implication, using the latter two and conjunction to construct the equivalence.

Conclusion

The described form of constructing a model of the logic of self-referential sentences reduces the many-valued of estimates to a two-valued logic, which corresponds to the spirit of Suzsko's sentences.

References

[1] Dynamics of one-dimensional mappings / A. N. Sharkovskii, S. F. Kolyada, A. G. Sivak, V. V. Fedorenko. — Kiev : Naukova Dumka, 1989. — 216 p. [In Russian]

[2] *Emily, M.* Pierce's Paradoxical Solution to the Liar's Paradox // Notre Dame Journal of Formal Logic. — 1975. — Vol. 16, №3. — P. 369–374.

[3] *Feferman, S.* Toward Useful Type-Free Theories I. // The Journal of Symbolic Logic. — 1984. — Vol. 49, №1. — P. 75–111.

[4] *Johnstone, A.* Self-reference, the Double Life and Gödel // Logique et Analyse. — 1981. — Vol. 24. — P. 35–47.

[5] *Priest, G.* The Logic of Paradox // Journal of Philosophical Logic. — 1979. — Vol. 8. — P. 219–241.

Библиографическая ссылка

Stepanov, V. A. In defense of the self-reference quantifier Sx. Approximation by dynamic systems. // Всероссийская научная конференция «Математические основы информатики и информационно-коммуникационных систем». Сборник трудов. — Тверь : ТвГУ, 2021. — С. 272–278.
https://doi.org/10.26456/mfcsics-21-38

Сведения об авторах

Vladimir Alekseevich Stepanov
Dorodnitsyn Computing Center of FIC CSC RAS. Researcher

Vavilov st. 40, 119333 Moscow, Russia
E-mail: vastvast@yandex.ru